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Stable equilibrium based on Lévy statistics: Stochastic collision models approach

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We investigate equilibrium properties of two very different stochastic collision models: (i) the Rayleigh particle and (ii) the driven Maxwell gas. For both models the equilibrium velocity distribution is a Lévy distribution, the Maxwell distribution being a special case. We show how these models are related to fractional kinetic equations. Our work demonstrates that a stable power-law equilibrium, which is independent of details of the underlying models, is a natural generalization of Maxwell's velocity distribution.

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There is a strong analogy between the Gaussian central limit theorem (GCLT) and the relaxation to thermal equilibrium of the Boltzmann equation (Ref. [1] and references therein). However, the GCLT is nonunique, which may imply that standard thermal equilibrium is nonunique. The Lévy central limit theorem (LCLT) considers the case of summation of independent identically distributed random variables with an *infinite* variance [2]. Hence following Montroll and Shlesinger it is natural to ask if generalized equilibrium concepts based on LCLT are meaningful [3]. Here we start answering this question using two very different types of collision models which still reveal the same type of equilibrium. We note that Lévy statistics has many physical applications [4], however, its possible relation to generalized forms of equilibrium statistical mechanics is an open field of research. Recently, Bobylev and Cercignani [1] investigated a nonlinear Boltzmann equation with an infinite velocity variance showing that the solution exists, and obtaining certain bounds on it. In Ref. [1] the possibility of a relation between solutions of the Boltzmann equation [5] and LCLT was briefly pointed out.

Our goal in this paper is to show that an alternative equilibrium concept naturally emerges from old stochastic collision models. Our models demonstrate that: (i) Lévy velocity distributions serve as the natural generalization of the Maxwell velocity distribution, (ii) generalized power-law equilibrium can be derived from kinetic models, *there is no need to postulate a specific form of power-law equilibrium,* and (iii) the Lévy equilibrium obtained here possesses a certain domain of attraction, is unique, and does not depend on certain details of the underlying models.

Model 1. We consider a one-dimensional tracer particle with the mass M coupled with gas particles of mass m. The tracer particle velocity is V_M. At random times the tracer particle collides with gas particles whose velocity is denoted with $\tilde{v}_m$. Collisions are elastic hence from conservation of momentum and energy $V_M^+=\xi_1 V_M^-+\xi_2\tilde{v}_m$, where $\xi_1=(1-\epsilon)/(1+\epsilon)$, $\xi_2=2\epsilon/(1+\epsilon)$, $\epsilon\equiv m/M$ is the mass ratio, and V_M^+ (V_M^-) is the velocity of the tracer particle after (before) a collision event. The duration of the collision events is much shorter than any other time scale in the problem. The collisions occur at a uniform rate R. The probability density function (PDF) of the gas particle velocity is $f(\tilde{v}_m)$. This PDF does not change during the collision process, indicating that re-collisions of the gas particles and the tracer particle are neglected.

Many works considered this type of model, imposing the condition that the gas particles are distributed according to Maxwell's law, i.e., $f(\tilde{v}_m)$ is Maxwellian. Since we are now investigating possible generalizations of Maxwell's law we change this strategy and assume that $f(\tilde{v}_m)$ is non-Maxwellian. The goal is to see when and how the tracer particle reaches a universal equilibrium, which does not depend on the detailed shape of $f(\tilde{v}_m)$.

We now consider the equation of motion for the tracer particle velocity PDF $W(V_M,t)$ with initial conditions concentrated on $V_M(0)$. Standard kinetic considerations yield the linear Boltzmann equation

$$\dot{W}(V_M,t)=-RW(V_M,T)+R\int_{-\infty}^{\infty}dV_M^-\int_{-\infty}^{\infty}d\tilde{v}_m \times W(V_M^-,t)f(\tilde{v}_m)\delta(V_M-\xi_1 V_M^--\xi_2\tilde{v}_m), \tag{1}$$

where the delta function gives the constraint on energy and momentum conservation in collision events. As usual, the first (second) term in Eq. (1) describes a tracer particle leaving (entering) the velocity point V_M at time t. Equation (1) contains convolution integrals in velocity space, hence we consider now its Fourier transform (FT). Let $\overline{W}(k,t)$ be the FT of the velocity PDF $W(V_M,t)$. Using Eq. (1), the equation of motion for $\overline{W}(k,t)$ is a finite difference nonlocal equation,

$$\dot{\overline{W}}(k,t)=-R\overline{W}(k,t)+R\overline{W}(k\xi_1,t)\overline{f}(k\xi_2), \tag{2}$$

where $\overline{f}(k)$ is the FT of $f(\tilde{v}_m)$. The solution of the equation of motion, Eq. (2), is obtained by iterations

$$\overline{W}(k,t)=\sum_{n=0}^{\infty}\frac{(Rt)^n\exp(-Rt)}{n!}e^{ikV_M(0)\xi_1^n}\prod_{i=1}^{n}\overline{f}(k\xi_1^{n-i}\xi_2). \tag{3}$$

This solution has a simple interpretation. The probability that the tracer particle has collided n times with the gas particles

is given according to the Poisson law $P_n(t)=[(Rt)^n/n!]\exp(-Rt)$. Let $W_n(V_M)$ be the PDF of the tracer particle conditioned that the particle experiences n collision events. It can be shown that the FT of $W_n(V_M)$ is $\overline{W}_n(k)=e^{ikV_M(0)\xi_1^n}\Pi_{i=1}^n\overline{f}(k\xi_1^{n-i}\xi_2)$. Thus Eq. (3) is a sum over the probability of having n collision events in time interval $(0,t)$ times the FT of the velocity PDF after exactly n collision event.

In the long-time limit $\overline{W}_{eq}(k)\equiv\lim_{t\to\infty}\overline{W}(k,t)$ an equilibrium is obtained from Eq. (3). We notice that when $Rt\to\infty$, $P_n(t)=(Rt)^n\exp(-Rt)/n!$ is peaked in the vicinity of $\langle n\rangle=Rt$, hence it is easy to see that

$$\overline{W}_{eq}(k)=\lim_{n\to\infty}\prod_{i=1}^{n}\overline{f}(k\xi_1^{n-i}\xi_2). \tag{4}$$

In what follows we investigate properties of this equilibrium. We note that similar equilibrium can be obtained also if the collision process is not described by the Poisson law, any $P_n(t)$ which is peaked on $n\to\infty$ when $t\to\infty$, with (nearly) zero support for finite values of n, will exhibit this behavior.

We will consider the Rayleigh weak collision limit $\epsilon\to 0$. This limit is important since number of collisions needed for the tracer particle to reach an equilibrium is very large. Hence in this case we may expect the emergence of a general equilibrium concept which is not sensitive to the precise details of the velocity PDF $f(\tilde{v}_m)$ of the gas particles.

We assume that statistical properties of gas particles velocities can be characterized with an energy scale T. Since T, m, and $\tilde{v}_m$ are the only variables describing the gas particle we have

$$f(\tilde{v}_m)=\frac{1}{\sqrt{T/m}}q\left(\frac{\tilde{v}_m}{\sqrt{T/m}}\right). \tag{5}$$

We also assume that $f(\tilde{v}_m)$ is an even function, as expected from symmetry. The dimensionless function $q(x)\geqslant 0$ satisfies a normalization condition $\int_{-\infty}^{\infty}q(x)dx=1$, otherwise it is rather general. The scaling assumption made in Eq. (5) is very natural, since the total energy of gas particles is nearly conserved.

We first consider the case where moments of $f(\tilde{v}_m)$ are finite. The second moment of the gas particle velocity is $\langle\tilde{v}_m^2\rangle=(T/m)\int_{-\infty}^{\infty}x^2q(x)dx$. Without loss of generality we set $\int_{-\infty}^{\infty}x^2q(x)dx=1$. The scaling behavior Eq. (5) yields $\langle\tilde{v}_m^{2n}\rangle=(T/m)^nq_{2n}$, where the moments of $q(x)$ are defined according to $q_{2n}=\int_{-\infty}^{\infty}x^{2n}q(x)dx$. Thus the small k expansion of the gas particle characteristic function is

$$\overline{f}(k)=1-\frac{Tk^2}{2m}+q_4\left(\frac{T}{m}\right)^2\frac{k^4}{4!}+O(k^6). \tag{6}$$

Inserting Eq. (6) in Eq. (4) we obtain

$$\ln[\overline{W}_{eq}(k)]=-\frac{T}{2m}g_2(\epsilon)k^2+\frac{q_4-3}{4!}\left(\frac{T}{m}\right)^2g_4(\epsilon)k^4+O(k^6), \tag{7}$$

where $g_n(\epsilon)=(2\epsilon)^n/[(1+\epsilon)^n-(1-\epsilon)^n]$. The interesting thing to notice is that in the limit $\epsilon\to 0$, the second term on the right hand side of Eq. (7) is zero, thus q_4 is an irrelevant parameter in the problem. In a similar way one can show that all terms in the expansion containing q_{2n} with $n>1$ vanishing in the Rayleigh limit $\epsilon\to 0$. Thus using Eq. (7)

$$\lim_{\epsilon\to 0}\ln[\overline{W}_{eq}(k)]=-\frac{T}{2M}k^2. \tag{8}$$

From Eq. (8) it is easy to see that the Maxwell velocity PDF for the tracer particle M is obtained. Thus as expected Maxwell's equilibrium is stable in the sense that for a large class of gas particle velocity PDF's Maxwell equilibrium is obtained.

Now we assume that $f(\tilde{v}_m)$ has a power-law behavior, i.e., $q(x)\propto|x|^{-(1+\alpha)}$ when $|x|\to\infty$ and $0<\alpha<2$. For this case the gas particle characteristic function is

$$\overline{f}(k)=1-\frac{q_\alpha}{\Gamma(1+\alpha)}\left(\frac{T}{m}\right)^{\alpha/2}|k|^\alpha+\frac{q_\beta}{\Gamma(1+\beta)}\left(\frac{T}{m}\right)^{\beta/2}|k|^\beta+o(|k|^\beta), \tag{9}$$

where $\alpha<\beta\leqslant 2\alpha$. q_α and q_β are dimensionless numbers which depend of course on $q(x)$. Without loss of generality we may set $q_\alpha=1$. In Eq. (9) we have used the assumption that $f(\tilde{v}_m)$ is even.

Inserting Eq. (9) in Eq. (4) we obtain a small k expansion $\ln[\overline{W}_{eq}(k)]$. Taking the limit $\epsilon\to 0$ one can show that the terms containing q_β are much smaller than the leading term $\ln[\overline{W}_{eq}(k)]\propto-|k|^\alpha$. Thus q_β, and in a similar way higher-order coefficients, become the irrelevant parameters of the problem. Thus we find that the tracer particle equilibrium characteristic function is

$$\overline{W}_{eq}(k)\sim\exp\left[-\frac{2^{\alpha-1}}{\alpha\Gamma(1+\alpha)}\left(\frac{T}{M}\right)^{\alpha/2}\frac{|k|^\alpha}{\epsilon^{1-\alpha/2}}\right], \tag{10}$$

thus a Lévy velocity distribution for the tracer particle is obtained. For $\alpha\neq 2$ the equilibrium Eq. (10) depends on ϵ, while for the Maxwell's case $\alpha=2$, the equilibrium is independent of the coupling constant ϵ. Equation (10) implies that variance of the velocity diverges when $\alpha<2$. For the non-Maxwellian case the velocity distribution is characterized by the scale $(T/M)^{\alpha/2}$ which determines the width of the velocity distribution.

The asymptotic behavior Eq. (10) is now demonstrated using numerical examples. We consider three types of gas particle velocity PDF's, for large values of $|v_m|\to\infty$ these PDF's exhibit $f(\tilde{v}_m)\propto|\tilde{v}_m|^{-5/2}$, namely $\alpha=3/2$. Case 1 $f(\tilde{v}_m)=N_1/[1+3^{2/3}\sqrt{m/(4T)}|\tilde{v}_m|]^{5/2}$, case 2, $f(\tilde{v}_m)=N_2/[1+m\tilde{v}_m^2/(0.439T)]^{5/4}$, where N_1 and N_2 are normal-

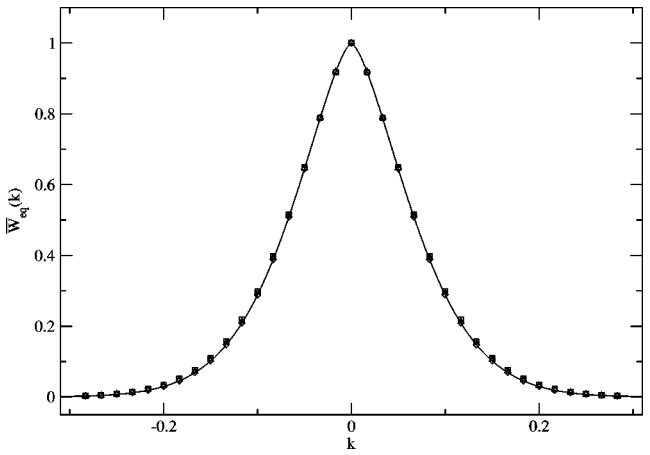

FIG. 1. We show the FT of the equilibrium velocity distribution of the tracer particle. Numerically exact solutions of the problem are obtained using three long tailed gas particle velocity PDF's defined in text: case 1 squares, case 2 triangles, and case 3 diamonds. The tracer particle equilibrium is well approximated by the Lévy distribution the solid curve; $\overline{W}_{eq}(k) \sim \exp(-2.211|k/\epsilon^{1/6}|^{3/2})$. For the numerical results we used $T=4.555$, $\epsilon=1e-5$, and $M=1$.

ization constants. Case 3, the gas particle velocity PDF, is a Lévy PDF with index 3/2 whose FT is $\overline{f}(k) = \exp[-(T/m)^{3/4}[|k|^{3/2}/\Gamma(5/2)]]$.

According to our theory these power-law velocity PDF's yield a Lévy equilibrium for the tracer particle when the mass ratio becomes small, Eq. (10). In Fig. 1 we show numerically exact solution of the problem for cases 1–3. These solutions, obtained using Eq. (4) for finite values of ϵ, show a good agreement between numerical results and the asymptotic theory. The Lévy equilibrium for the tracer particle is not sensitive to precise shape of the velocity distribution of the gas particle, and hence like the Maxwell distribution is stable.

We now consider a Fokker-Planck equation which describes the evolution of the tracer particle PDF $W(V_m,t)$ towards the Lévy equilibrium Eq. (10). The equation is of fractional order and is obtained using a small ϵ expansion of Eq. (1) (details to be published),

$$\frac{\partial W(V_M,t)}{\partial t} \simeq \frac{\overline{D}}{\epsilon^{1-\alpha/2}} \frac{\partial^\alpha W(V_M,t)}{\partial |V_M|^\alpha} + \gamma \frac{\partial}{\partial V_M}[V_M W(V_M,t)]. \tag{11}$$

In Eq. (11) the Riesz fractional derivative was used [4], and the dissipation term is $\gamma = 2\epsilon R$. A generalized Einstein relation,

$$\overline{D} = \frac{2^{\alpha-1}}{\Gamma(1+\alpha)} \left(\frac{T}{M}\right)^{\alpha/2} \gamma, \tag{12}$$

yields the relation between the transport coefficients $\overline{D}$ and γ. When $\alpha=2$ the Einstein relation is recovered. Note that Refs. [6–9] investigated related fractional processes based on a stochastic approach (e.g., Langevin equations with Lévy noise). In those investigations dissipation and fluctuations were treated as though they are independent, hence the equilibrium obtained there differs from ours.

Model 2. The question remaining is whether Lévy equilibrium a general feature, which might be obtained from other collision models. Specifically, it is interesting to see if Lévy equilibrium is compatible with a nonlinear Boltzmann equation approach, since one may suspect that the Lévy behavior obtained so far is limited to linear Boltzmann models. For this aim we investigated the one-dimensional (1D) driven inelastic Maxwell model (DIMM). This model was investigated extensively in recent years in the context of inelastic gases assuming finite variance boundary conditions (see details below) [10,11]. Our goal is to investigate DIMM in the quasielastic limit showing that Lévy statistics describes the equilibrium; the Maxwell-Gauss distribution is recovered in the proper limit.

First consider the inelastic Maxwell model in the *absence* of external driving forces $\dot{W}(V,t) = I(V,W)$, where the nonlinear collision integral is

$$I(V,W) \equiv -W(V,t) + \frac{1}{p}\int_{-\infty}^{\infty} W(u,t) W\left(\frac{v-qu}{p},t\right) du. \tag{13}$$

In Eq. (13) $p=(r+1)/2$ and $q=1-p$ where r is the restitution coefficient $0<r\leqslant 1$. The kinetic scheme describes a situation where momentum is conserved during collision events, while energy is conserved only when $r=1$. If $r<1$ the steady state solution of Eq. (13) is $W_{ss}(V)=\delta(V)$, reflecting the loss of energy during collision events. Note that for elastic collisions $r=1$, any initial velocity distribution is a steady-state solution. This is expected (and not informative) since two identical 1D elastic particles exchange their velocities in collision events.

Let $\overline{W}(k,t)$ be the FT of $W(V,t)$. The boundary conditions we will consider are

$$\overline{W}(k,t) \sim 1 - \frac{\langle |V|^\alpha \rangle |k|^\alpha}{\Gamma(1+\alpha)}, \tag{14}$$

for small k. Since $W(V,t)$ is a non-negative PDF we have $0<\alpha\leqslant 2$. Using the Boltzmann equation (13) it is easy to show that in the elastic limit $r=1$, $\partial\langle |V|^\alpha\rangle/\partial t = 0$. For the standard case $\alpha=2$ considered in Ref. [11], Eq. (14) simply reflects energy conservation, i.e., $\langle V^2\rangle$ is a constant of motion. For $\alpha<2$, $\langle |V|^\alpha\rangle$ describes the width of the probability packet, which for elastic collision is a conserved quantity.

As mentioned, when $r<1$ the inelastic collisions will shrink any initial probability packet to be concentrated on $V=0$. Similar to previous work [11], an *infinitesimal* heating term is added to the equation of motion, which compensates the energy loss. We will consider the boundary conditions described in Eq. (14), while Ref. [11] considered the Gaussian case $\alpha=2$. To obtain behavior compatible with Eq. (14) we consider the fractional DIMM,

$$\frac{\partial W(V,t)}{\partial t} - D_\alpha \frac{\partial^\alpha W(V,t)}{\partial |V|^\alpha} = I(V,W). \tag{15}$$

It is more convenient to consider this fractional equation in Fourier space; this yields the nonlinear and nonlocal equation

$$\dot{\overline{W}}(k,t) + (1 + D_\alpha |k|^\alpha)\overline{W}(k,t) = \overline{W}(pk,t)\overline{W}(qk,t). \tag{16}$$

For our aim this equation gives the definition of the fractional derivative in Eq. (15). Our aim is to investigate the steady-state solution of this equation in the quasielastic limit when $D_\alpha \to 0$ and $r \to 1$. This limit is taken in such a way that the boundary condition Eq. (14) is satisfied.

Using the condition $\partial \langle |V|^\alpha \rangle / \partial t = 0$, and Eqs. (14) and (16) we obtain

$$D_\alpha = \frac{\langle |V|^\alpha \rangle}{\Gamma(1+\alpha)}(1 - p^\alpha - q^\alpha). \tag{17}$$

Without loss of generality we may set now $\langle |V|^\alpha \rangle = 1$. For $\alpha = 1$ we have $D_\alpha = 0$, while for $0 < \alpha < 1$ D_α obtains negative values. The case $\alpha = 1$ marks the transition between a finite ($\alpha > 1$) and infinite ($\alpha < 1$) first-order moment of velocity $\int_{-\infty}^{\infty} |V| W(V,t) dV$. For $\alpha < 1$ no steady state is obtained, since the dissipation due to collisions is not strong enough to compensate the heating. Our results in what follows are restricted to $1 < \alpha \leqslant 2$.

Steady-state solution of Eq. (16) satisfies

$$(1 + D_\alpha |k|^\alpha)\overline{W}_{ss}(k) = \overline{W}_{ss}(pk)\overline{W}_{ss}(qk). \tag{18}$$

An iteration method is used to obtain the solution. Let $\psi(k) \equiv \ln[\overline{W}_{ss}(k)]$, and using Eq. (18) we have

$$\psi(k) = \psi(pk) + \psi(qk) - \ln[1 + D_\alpha |k|^\alpha]. \tag{19}$$

The boundary condition Eq. (14) yields $\psi(k) \sim -|k|^\alpha / \Gamma(1+\alpha)$. The solution of Eq. (19) is obtained using the iteration rule

$$\psi_{n+1}(k) = \psi_n(pk) + \psi_n(qk) - \ln[1 + D_\alpha |k|^\alpha], \tag{20}$$

where $\lim_{n\to\infty} \psi_n(k) = \psi(k)$ and the "initial condition" is $\psi_0(k) = -\ln[1 + D_\alpha |k|^\alpha]$. Using these rules and some algebra involving series expansions, we find

$$\psi(k) = -\sum_{n=1}^{\infty} \frac{(-1)^{n+1} |k|^{\alpha n} D_\alpha^n}{n(1 - q^{\alpha n} - p^{\alpha n})}, \tag{21}$$

where the condition $1 < \alpha \leqslant 2$ was used. Inserting Eq. (17) in Eq. (21) we obtain

$$\psi(k) = -\frac{|k|^\alpha}{\Gamma(1+\alpha)} + \frac{\alpha(1-r)}{8} \frac{|k|^{2\alpha}}{[\Gamma(1+\alpha)]^2} + O(|k|^{3\alpha}). \tag{22}$$

The interesting thing to notice is that the second term on the right-hand side of Eq. (22) vanishes when the elastic limit $r \to 1$ is considered. Inserting Eq. (17) in Eq. (21) one can show that in the elastic limit $\lim_{r\to 1} \psi(k) = -|k|^\alpha / \Gamma(1+\alpha)$. Hence the steady-state characteristic function is a stretched exponential,

$$\lim_{r\to 1} \overline{W}_{ss}(k) = \exp\left[-\frac{|k|^\alpha}{\Gamma(1+\alpha)} \right]; \tag{23}$$

the inverse Fourier transform (FT) of this equation yields the symmetric stable Lévy density [2]. It is rewarding to find that Maxwell and Lévy equilibriums are obtained only in the elastic limit, thus conservation of energy in the collision events is related to the Lévy-Maxwell behavior. Far from this limit results not directly related to the Gauss-Lévy central limit theorem are obtained, Eq. (21).

To conclude, we demonstrated the relation between equilibrium properties of very different types of collision models and Lévy statistics. Thus stable behavior transcends details of individual models and hence, I suspect, can be found in other types of collision models. To support the idea that Lévy velocity distribution might be found in other models, I note the interesting work of Min *et al.* [12] who used numerical simulations of a long-range interacting vortex model, and showed that distribution of velocity fields are Lévy stable.

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